Gumbel Softmax Estimation for Binary Random Variables

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In this script, we will show that given a normalized Bernoulli logit $\alpha = [p, 1-p]$, the corresponding Gumbel softmax estimation is

$$\sigma\left\{-\frac{1}{\tau}\left(\log\frac{p}{1-p} + \log\frac{u}{1-u}\right)\right\}, \tag{1}$$

where $\sigma(x) = \frac{1}{1+e^{-x}}$ is a sigmoid function, τ is a temperature coefficient, $u \sim \text{Uniform}(0,1)$. It turns out that $p(\lim_{\tau\to 0} x = 1) = p$. Similarly, if given an unnormalized Bernoulli logit $\alpha = [\alpha_1, \alpha_2]$, the corresponding Gumbel softmax estimation is

$$\sigma\left\{-\frac{1}{\tau}\left(\log\frac{\alpha_1}{\alpha_1+\alpha_2} + \log\frac{u}{1-u}\right)\right\}. \tag{2}$$

It turns out that $p(\lim_{\tau\to 0} x = 1) = \frac{\alpha_1}{\alpha_1+\alpha_2}$.

Let's first recall the Gumbel softmax [1] estimation for Category random variables. Given am unnormalized categorical logits $\alpha = [\aleph_1, \ldots, \alpha_n]$, the Gumbel softmax estimation is

$$\frac{\exp\{(\log\alpha_k + g_k)/\tau\}}{\sum_i \exp\{(\log\alpha_i + g_i)/\tau\}}, \tag{3}$$

where $g = [g_1, \ldots, g_n]$ is standard Gumbel random variables. It turns out that $p(\lim_{\tau\to 0} x_k = 1) = \frac{\alpha_k}{\sum_i \alpha_i}$.

Next, it is helpful to know that the difference of two independent Gumbel random variables $X_1 \sim \text{Gumbel}(a_1), X_2 \sim \text{Gumbel}(a_2)$ follows a Logistic distribution, *i.e.*, $X_1 - X_2 \sim \text{Logistic}(\alpha_1 - \alpha_2)$ (see section 1.2 in my another note for details). Moreover, sampling from a standard logistic distribution can be done by $x \sim \text{Logistic}(1) \Leftrightarrow x = \log\frac{u}{1-u}, u \sim \text{Uniform}(0,1)$. Now, we establish sufficient prerequisites to prove Gumbel softmax estimation for the binary case. Specifically,

$$\begin{aligned}
\text{Binary Gumbel Softmax} &= \frac{\exp\{(\log p + g_1)/\tau\}}{\exp\exp\{(\log p + g_1)/\tau\} + \exp\exp\{(\log(1-p) + g_2)/\tau\}} \\
&= \frac{1}{1 + \exp\{(\log(1-p) - \log p + g_2 - g_1)/\tau\}} \\
&= \frac{1}{1 + \exp\{(\log(1-p) - \log p + \log u - \log(1-u))/\tau\}} \\
&= \sigma\left\{-\frac{1}{\tau}\left(\log\frac{p}{1-p} + \log\frac{u}{1-u}\right)\right\}.
\end{aligned}$$

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References

[1] Eric Jang, Shixiang Gu, and Ben Poole. Categorical reparameterization with gumbel-softmax. *arXiv preprint arXiv:1611.01144*, 2016.